1 Divination

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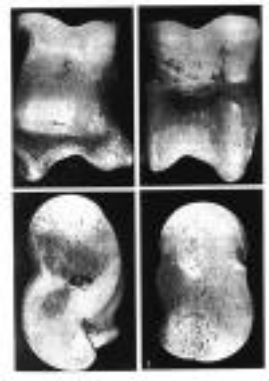
1 Ancient Games



The astragalus is a small bone, about an inch cube, found in the heel of mammals just above the heel-bone or talus. It is used even today in childrens' games.

Astragali were used in board games in the First Dynasty in Egypt, c 3500 B.C.; archaeological evidence consists of boards, counters, and astragali for various games, including one similar to Snakes and Ladders. The astragali may have been first used as counters, then adapted to introduce a random element.

In throwing an astragalus, only four faces are possible outcomes, and the asymmetry of the bone makes the probabilities of the four faces quite different. A standard game in Roman times used four astragali. A regular six sided die may have developed originally by shaving the astragalus suitably; such dice are found in 3000bc in Iraq and India; the modern die with opposite faces summing to seven was settled by 1400bc. The astragali continued to be used in classical times for gambling, and are still used by some American Indian tribes.

2 Divination

In every religion, one must attempt to determine the intentions of the God or Gods, and often the Gods seem unwilling to reveal their intentions. *Divination* is determining God's will. When divination is done by mechanical means, it is called *sortilege.*

The most common early form of sortilege is the drawing of lots. (" Lot" is a word of German origin, from which the Italian "lottery" is derived, and English borrowed them both later. The real estate "lot" comes from the land of the deceased being divided up by lot into lots.) In primitive tribes, the person guilty of some offence is found by all drawing from a set of marked stones or pieces of wood or straws of unequal length. Since the God knows the awful punishment awaiting the guilty, the God will take care that the real guilty person gets the bad lot.

From David, p 14, a quotation from the journal of John Wesley 4 March 1737:
Mr Wesley is trying to decide whether or not he should marry.
Having both of us [Mr Delamotte and himself] sought God by deep consideration, fasting and prayer, in the afternoon we conferred together but could not come to any decision. We both apprehended Mr. Ingham's objection to be the strongest, the doubt whether she was what she appeared. But this doubt was too hard for us to solve. At length we agreed to appeal to the Searcher of Hearts. I accordingly made three lots. In one was writ "Marry"; in the second " Think not of it this year". After we had prayed to God to "give a perfect lot", Mr Delamotte drew the third, in which the words were " Think of it no more".Instead of the agony I had reason to expect, I was enabled to say cheerfully " Thy will be done". We cast lots again in order to know whether I ought to converse with her any more, and the direction I received from God was " Only in the presence of Mr. Delamotte".

The game of odds and evens has been used for divination from the oldest times. The priest holds a large number of pebbles or grain; a random number is poured on the ground; if the number is odd, the question is answered yes, and if the number is even the question is answered no. You can do this with a dandelion puffball, blowing the seed carriers off one by one and saying "*she loves me*", "*she loves me not*" in turn, till you find out which it is.

Again from David, p25, a quote from Cicero(De Divinatione):
Do you really feel that lots require any discussion? What is a lot anyway? It belongs virtually in the same category as " guess-the-fingers", knucklebones, and dice. In all these games audacity and luck win, not reason and thought. As a matter of fact the whole system of peering into the future by means of lots was the invention of tricksters who were only interested in their own financial welfare, or in fostering superstition and folly.

3 From Hasofer , Random mechanisms in the Talmud

2. The Jewish attitude to chance mechanisms

As with many other matters connected with ethics and religion, the Jewish attitude to the use of chance mechanisms in the ancient world was diametrically opposite to that of neighbouring nations. While dice gambling was in great favour among Greeks and Romans, it was forbidden among Jews, and persons indulging in it were subject to various legal disabilities. Again, astragali and dice were freely used for divination in temples devoted to idol worship, a practice strictly forbidden under penalty of death among the Jews (Deut. xviii. 9–14).

On the other hand, random mechanisms were used extensively in religious ceremonies, as well as for various legal purposes. The main technique used was that of drawing lots out of an *urn*. But other methods were also used, as we shall see in the sequel.

There were two main ideas underlying the use of lots. The first one was that the use of lots was a fair method of allocating duties or rewards among various contenders. This idea is clearly expressed in Prov. xviii. 18 'The lot causeth disputes to cease, and it decideth between the mighty'. According to Rashi's Commentary of the Bible, the Hebrew word translated by 'mighty' actually means people who have a mighty quarrel between them. Thus the meaning of the second part of the proverb is that the lot separates even those engaged in the mightiest quarrel. But we shall see later that the only methods of drawing lots which were in use had strictly equiprobable outcomes. Thus it seems that the notion of a 'fair game' was quite familiar to the Rabbis.

The second main idea underlying the use of lots is that the result of a lot undertaken by the Commandment of G'd will in fact give expression to G'd's will. This idea is most clearly expressed in the controversy between Joshua and Achan described in Sanh. 43*b*. The background of the event is to be found in Joshua vii. 1–26.

We read:

'When the Holy One, blessed be He, said to Joshua, "Israel hath sinned", he asked Him, "Sovereign of the Universe, who hath sinned?" "Am I an informer?" He answered, "Go and cast lots". Thereupon he went and cast lots, and the lot fell upon Achan. Said he to him "Joshua, doest thou convict me by a mere lot? Thou and Eleazar the Priest are the two greatest men of the generation, yet were I to cast lots upon you, the lot might fall on one of you". "I beg thee", he replied, "cast no aspersions on [the efficacy of] lots, for Eretz Yisrael is yet to be divided by means of lots, as it is written, *The land shall be divided by lot*"' (Num. xxvi. 55) (Sanh. 43*b*).

From this passage, it appears that the concept of blind chance had been considered and consciously rejected in favour of the notion of complete control of the results of lot-drawing by Divine Providence, at least in those cases where the drawing of lots was done by command of G'd.

3. Some examples of the use of random mechanisms

We shall now describe various random mechanisms encountered in Talmudic literature. Some were used only once, like the method of division of Israel between the tribes, and others were used regularly, like the method of allocation of daily chores to the priests in the Temple.

(i) *The division of Israel*

The main passage describing this event is found in Baba Bathra, 122*a*:

'Eleazar was wearing the Urim and Tummim, while Joshua and all Israel stood before him. An urn [containing the names] of the [twelve] tribes, and an urn [containing descriptions] of the boundaries were placed before him. Animated by the Holy Spirit, he gave directions, exclaiming: "Zebulun" is coming up and the boundary lines of Acco are coming up with it. [Thereupon] he shook well the urn of the tribes and Zebulun came up in his hand. [Likewise] he shook well the urn of the boundaries and the boundary lines of Acco came up in his hand. Animated again by the Holy Spirit, he gave directions, exclaiming: "Naphtali" is coming up and the boundary lines of Gennesar are coming up with it. [Thereupon] he shook well the urn of the tribes and Naphtali came up in his hand. And [so was the procedure with] every [other] tribe.'

It is interesting to note here that the results of the drawing were announced before the drawing in order to emphasize that the lot was an expression of Divine will. Rabbi Samuel ben Meir (1085–1158) in this commentary to this section of Baba Bathra writes:

'They were asking the Urim and Tummim first, before the appointed person drew from the urn, in order that the minds of the Israelites should cool off, seeing that the lot came up as prophesied, and they would thus be convinced that the division was honest.'

Let us remark that the method of lot-drawing was not a direct method, but a more elaborate coincidence method, providing an additional safeguard against cheating. In fact, according to the Jerusalem Talmud (Yoma, Chap. 4, Sect. 1) the drawing from the two urns was performed by two different priests.

(ii) *The drawing of lots for the scapegoat on the day of atonement*

Every year, on the Day of Atonement, the High Priest used to sacrifice two he-goats in the Temple of Jerusalem, in accordance with Lev. xvi. 5–10.

'And from the congregation of the children of Israel shall he (Aaron) take two goats for a sin-offering, and one ram for a burnt offering . . . . And he shall take the two goats, and place them before the Lord at the door of the tabernacle of the congregation. And Aaron shall put lots upon the two goats; one lot "for the Lord" and the other lot "for Azazel". And Aaron shall bring near the goat upon which fell the lot "for the Lord", and offer him for a sin-offering. But the goat on which fell the lot "for Azazel', shall be placed alive before the Lord, to make an atonement with him, by sending him away to Azazel into the wilderness.'

The details of this ceremony are discussed at great length in the tractate Yoma (37 *a*–41 *a*), but the account given there is not a connected one and is intermingled with discussions on other topics. So instead of giving the original Talmudic text, we shall quote the account given in the Mishneh Torah of Maimonides (Day of Atonement, Chap. 3). Every detail of this account is of course based on the Talmudic text, even though the sources are not indicated. We shall do the same later on for the details of the lot-drawing for the daily sacrifice.

Maimonides writes:

'Concerning the two lots: on one of them was written "for the Lord", and on the other was written "for Azazel". They might be made of any material: wood, stone or metal. However, one was not to be large and the other small, or one of silver and the other of gold. Rather, both were to be alike. They were, originally, made of wood, but in the Second Temple they were made of gold. Both lots were placed in a vessel that could contain two hands, so that one might put in both his hands without reaching purposely (for a particular lot). This vessel was unhallowed. It was made of wood and was called "the urn". . . .'

'The High Priest shook the urn and brought up in his two hands the two lots for the two he-goats. He then put the two lots upon the two animals, the one in his right hand on the animal on the right and the one in his left hand on the animal on the left. . . .'

Let us note that the Talmud explains that it was necessary to shake the urn, lest the high priest take one lot intentionally (Yoma 39*a*). For it was considered a happy omen when the lot for the Lord came up in the right hand, and the temptation was great to improve upon chance by dexterous manipulation.

An additional precaution was the fact that the two lots were not just inscribed, but actually engraved in order that the writing should not rub off (Jerusalem Talmud, Yoma, Chap. 4).

(iii) *The allocation of daily duties in the temple*

A completely different type of random mechanism, used for a very different purpose, namely the allocation of daily duties in the temple, is described in Yoma 22*a*. It is important to note that while in the two preceding ceremonies the drawing of lots was called 'Goral', which means a little ball or stone (Jastrow, 1950), the allocation of daily duties is called 'Payis', a word whose root verb means 'to pacify'. Thus the purpose of the allocation by lots seems to have clearly been the achievement of a fair and honest division. The fact that it was not carried out by the High Priest indicates that it was not primarily an appeal for G'd's decision.

We shall here also quote the description given in the Mishneh Torah (Daily Offerings, Chap. 4):

'How was the lot cast? The priests would stand in a circle and agree upon a number, say eighty, or a hundred or a thousand, or any other number upon which they would agree. The officer would say to them, "Show your fingers", and they would thrust out their fingers, one or two. If perchance someone thrust out three fingers, it was still counted. The thumb, however, might not be thrust out in the Sanctuary because of cheaters; the thumb being short, it might easily be thrust out and bent in. If one, therefore, did thrust out a thumb it was not counted. The officer would begin to count from the designated person whose mitre he had removed at first. He would count on the fingers and go round and round until he reached the number upon which they had agreed. The person at whose finger the number was reached would come out first in the lot for the service.'

'Why was the number agreed upon counted on the thrust-out fingers and not on the persons themselves? Because it was forbidden to count Israelites except by means of some other object; for it is said: "and he numbered them by sheep" (1 Sam. xv. 4)'

There are several points to be clarified in connexion with this passage. First of all the impression is gained that the officer was counting the individual fingers. This would of course make the lot asymmetrical, and give a better chance to those who thrust more fingers than others.

It is clear that here is an acid test of whether the Rabbis understood the concept of equiprobable outcomes or not. Moreover, as the lot-drawing was repeated day after day, and taking into account the quarrelsome temperament of the priests, which is stressed many times in the Talmud, and their deep personal involvement in the outcome of the lot-drawing, it is reasonable to infer that any departure from an equitable distribution of the chores would have been noticed. Thus the law of large numbers is also involved in clarifying the exact method of counting used.

The Talmud discusses the matter and unequivocally states that *all* the fingers of one person were counted for *one*. Here is the quotation:

'*And how many did they put forth? One or two.* If they may put forth two, why is it necessary to mention that they may put forth one?—R. Hisda said: This is no difficulty: The one speaks of healthy persons, the other of sick ones. Thus it has been taught: One finger is put forth, but not two. To whom does this rule apply? To a healthy person, but a sick one may put forth even two. But the Yehidim (i.e. the scholars) put forward two and *one counts only one thereof*. But has it not been taught: One does not put forth either the third finger or the thumb because of tricksters, and if one had put forth the third finger it would be counted, but if one had put forth the thumb it would not be counted, and not alone that but the officer strikes him with a *pekia* (i.e. a whip)?—What does "it would be counted" mean? *Only one*' (Yoma 23*a*). Normally only one finger was lifted. But if there were present some older, weaker or sick priests for whom it was inconvenient to put one finger forth and hold it aloft until the count was over, the officer would require all to put forth two fingers, which is less of an effort.

The theoretical basis of the method is of course that the sum modulo m of a random variable X equidistributed on the numbers $1, \ldots, m$ and any random variable on the integers independent of X is equidistributed on $1, \ldots, m$.

The reason why putting forth the thumb was forbidden was that a trickster, foreseeing at the end of the count where it would end, might place his index-finger at some distance from the thumb, so that the officer would count his two fingers as belonging to two people, with the result that the count would be wrong and designed to serve the trickster's end.

It is worthwhile to mention that the number agreed upon for the counting had to be much larger than the number of priests present (Yoma 22*a*, Rashi's commentary). This eliminated any practical possibility of cheating, as the priests were very unlikely to carry out in their heads a division operation involving large numbers, and ensured the independence of the decisions of the priests and the counting officer.

Finally, it is interesting to note that in older times there used to be a race up the ramp of the Altar, and he that came first within four cubits of the Altar secured the task of clearing the ashes. Only if two of the participants were equal did they use lots. But it once happened that two were equal, and one of them pushed the other so that he fell and his leg was broken; and when the Court saw that they incurred danger, they ordained that they should not clear the Altar save by lots (Yoma 22*a*).

This confirms our statement that in this case the use of the lot was designed to prevent quarrels, i.e. to 'pacify' the participants in the allocation of the daily chore. Let us also note here that with respect to the burning of the Incense, a highly valued duty, sampling without replacement was used in preference to the usual sampling with replacement, so as to ensure that all new priests take a turn before a new round started.

(iv) *Lot-casting for the sacrifices on festivals*

A different technique of lot-casting was used to divide the meat of sacrifices between the priests on festivals. Details are given to the Commentary on the Mishnah by Maimonides (Shabbat 148*b*). Each priest participating in the drawing gave some object belonging to him. They then called some outside person and asked him to put one object on each portion. Each priest then took the portion on which the object belonging to him had been deposed. It is explicitly stated that the object of the lot-casting was to avoid quarrels between priests (Shabbat 149*b*). This drawing was called '*Halashim*'.

4. The Jewish attitude to dice gambling

The Talmudic word for dice is 'Kubia', from the Greek *κυβια*, itself a derivative of *κυβοσ*, a cube. Dice gambling has, however, a wider meaning in the Talmud. Thus we read in Sanhedrin 25*b*: 'Dice players include the following: Those who play with *pispasim* (i.e. polished blocks or stones), and not only with *pispasim*, but even with nut-shells and pomegranate peels.'

The Jewish attitude to dice playing revolves around the concept of *Asmakhta* (literally *reliance*). An *Asmakhta* denotes a promise to pay on fulfilment of a condition which the contracting party expects not to be fulfilled. According to some teachers, an *Asmakhta* is not a valid obligation (see Baba Bathra 168*a* and Baba Metsia 66*a*–66*b*).

We read in Sanhedrin 24*b*:

'*Mishnah:* And these are ineligible [to be witnesses or judges]: a gambler with dice, a usurer, a pigeon-trainer, and traders [in the produce] of the sabbatical year . . . R. Judah said: When is this so?—If they have no other occupation but this, but if they have other means of livelihood, they are eligible.

Gemara: What [wrong] does the dice player do? Rammi ben Hama said: [He is disqualified] because it (i.e. gambling) is an *Asmakhta*, and an *Asmakhta* is not legally binding.

R. Shesheth said: Such cases do not come under the category of *Asmakhta*; but the reason is that they (i.e. dice players) are not concerned with the general welfare (i.e. they do not contribute to the welfare of civilised society).'

The reason for the condemnation of gambling was that the gambler always expects to win, and therefore to get something for nothing. This the Rabbis considered immoral and akin to robbery. Robbery and dice-gambling are actually coupled in many disqualifications in the Talmud, e.g. Shebuoth 47*a* where it is explained that the dice gambler is disqualified only by the Rabbis, and not by the Torah, since he is a robber, but does not use violence.

On the other hand, the opinion of R. Shesheth that dice gambling is not an *Asmakhta* is explained as follows by the Tosafist Rabbenu Tam (1100–71). '[Dice gambling] is not an *Asmakhta* because, since there are two [players], each one conveys possession [of the stake] to the other, on the understanding that if he wins, the other one will convey possession [of the stake] to him; it is for the very enjoyment [of this prerogative] that he agrees to convey possession to his opponent' (Tosafot Sanhedrin 25*a*).

But even if the Rabbis admitted that at least some gambers adopted this more sober view of their chances of winning, they nevertheless condemned gambling as antisocial in as far as the Jewish society of the time was concerned. Gambling (and for that matter, even games not involving chance) was considered a waste of time, turning the Jews away from their prime duty: to study and practice the Torah. Thus we read in Kiddushin 21*b*: 'Rabbi Nahman said to Rabbi Anan: When you were at Meir's academy you wasted your time playing *Iskumandri*'. This is variously interpreted as meaning a game similar to chess or checkers, or dog racing.

That the use of chance mechanisms as such was not condemned, but only the intent to win, is shown by the fact that it was permitted to draw lots for the various portions at the Sabbath table between members of the one family (Shabbath 148*b*):

'[On the Sabbath] . . . a man may cast lots with his sons and the members of his household for the table (i.e. which portion of the food shall belong to each), provided that he does not intend to offset a large portion against a small one (i.e. all portions must be alike in size) . . .'

The use of lots in civil law

Only one application of lot-drawing to civil law is mentioned in the Talmud, namely in the division of an estate between brothers. We read 'It was taught: Rabbi Jose said: When brothers divide [an estate] (into equal shares), all of them acquire possession [of their respective shares] as soon as the lot for one of them is drawn. On what ground [is possession acquired]?—Rabbi Eleazar said: [Possession is acquired in the same way] as [at] the beginning of [the settlement of] the land of Israel. As [at that] beginning [the acquisition was] by lot, so here [also it is] by lot. Since then, however, [the division was made] through the urn, and the *Urim and Tummim*, [should not the division] here also [be made] through the urn and the *Urim and Tummim*?—Rabbi Ashi replied: [The lot alone suffices here] because [in return for] the benefit of mutual agreement they determine to allow each other to acquire possession [by the lot alone]' (Baba Bathra 106*b*).

There must, however, have been many more uses of lot-drawing, as is indicated by Prov. xviii. 18 quoted above.

4 Why No Probability before Pascal?

Hacking discusses various proposed explanations for the lack of a formal probability calculus in the ancient world, even though there was much use of random mechanisms in divination and gambling.

(1) Belief in determinism prevented any thought about randomness. But Hacking objects, the belief in determinism came from the success of mechanistic Newtonian laws about the same time as the development of probability, so this cannot be the reason for the late development of probability.

(2) Lots and dice express of the will of the gods, and it would be impious to try to compute the outcomes. But Hacking objects, there were plenty of irreligious gamblers in classical times who gambled without any intent to find the will of the gods.

(3) It is difficult to understand probability without simple equal probability alternatives as examples, and the astragali are unsymmetrical. But says Hacking there were plenty of very good dice and equal probability lot mechanisms.

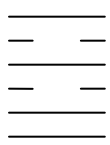
(4) There was no economic incentive to know and manipulate exact probabilities until Pascal. Hacking produces some earlier economic needs.

(5) People couldn't do the arithmetic until positional Indian arithmetic (via the arabs) became commonplace in Europe. There is some evidence for this. The names *hazard* and *algebra* are of Arabic origin, and the first probability calculations by the Italians may well have been borrowed from the arabs. In addition, dicing has been used for thousands of years in India, and there is some evidence of probability calculations in the ancient Indian literature.

5 The I Ching

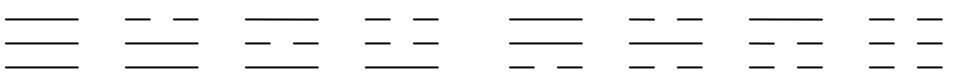
The I Ching originated in the Xia dynasty (2005-1766bc), but its present form was set with the commentaries of Confucius in the Zhou dynasty (1122-221bc).
The I Ching is composed of 64 *gua* that are selected at random; it used to be done with Yarrow stalks. The first occurrence of combinatorics per se may be in the 64 hexagrams of the *I Ching*. (However, the more modern binary ordering of these is first seen in China in the 10th century.) The modern masters use coins or cards.

Each gua makes predictions and assertions about the present situation of the user. A typical gua is represented by a 6 line symbol such as



Each line is either solid or broken, making 64 possible gua.
The gua has two names from the primary gua, which describe the possible eight values of the first three lines; the same names apply to the possible eight values of the second three lines.

The 8 primary gua:



Qian	Dui	Li	Zhen	Xun	Kan	Gen	Kun
Heaven	Lake	Fire	Thunder	Wind	Water	Mountain	Earth

Thus gua 31 is Lake over Mountain, Mutual Influence.
If you select that, then you should act according to the following poem, and its many subsidiary commentaries:

Mutual Influence.
Prosperous and smooth.
Favourable to be steadfast and upright.
Take a maiden as a wife.
Good fortune.

There is another feature, the moving line; in the older versions any line could be moving line or not, which allows for 4 possibilities for each line, 4096 possibilities altogether. In more recent versions, exactly one line is a moving line, producing 384

possible outcomes. You pay special attention to the text for the moving line, and to the gua obtained by changing the parity of the line.

There are two popular tools to cast the I Ching: the Yarrow Sticks and the Coins. However, these two tools are not equal, and can yield very different results based on numerical probability and statistics..

The third and most ancient method, that of heating a tortoise shell until it cracked, is long lost to us.

The three coins oracle became popular after its use by Shao Yun in the Sung dynasty (1127-1279). Each line was decided by tossing three coins. All faces gave a moving Yin(broken line), all backs a moving Yang(solid line), two faces a non-moving Yang, two backs a non-moving Yin. So the line is moving with probability 1/3.

The older, time consuming, and more interesting method of generating the lines is the Yarrow Stalk method. The original yarrow method has been lost, but it was reconstructed, at about the same time as the coin method was gaining in popularity, by Chu Hsi. In many I Ching books, you will find the yarrow method referred to as the 'authentic', original way to consult the I Ching. While its roots do go back further – to long before the invention of coinage in China – we have no way of knowing whether the method we use now is the original

Each line is generated in the same way beginning with 50 Yarrow stalks.
(1) drop 1 stalk
(2) divide the remainder into 2 bundles at random (it seems that empty groups are not permitted)
(3) drop a stalk from the right bundle
(4) put aside stalks from each bundle in sets of 4, until 4 or less stalks remain in each
(5) The number of stalks set aside from both bundles is 40 or 44.
(6)Repeat steps 1 through 4 getting 32 36 or 40 stalks. Then repeat again getting 24 28 32 or 36 stalks.
(7) 24 is a moving Yin, 28 is a stable Yang, 32 is a stable Yin, 36 is a moving Yang.

This procedure is repeated for each line of the I Ching. Compare the probabilities with the three coins oracle.